The 14th ISAV2024 International Conference on Acoustics and Vibration
11-12 Dec 2024 Karaj - Iran

# Nonlinear vibration analysis of electret-based energy harvesting beams

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## Abstract

This paper investigates the nonlinear dynamics of a capacitive electret-based capacitive energy harvester, focusing on solving the governing equations using semi-analytical methods, particularly the method of multiple scales (MMS). The harvester features a cantilever micro-beam above an electret layer, where surface voltage induces nonlinear electrostatic forces. These forces are simplified using a third-order Taylor expansion to reduce complexity while preserving key nonlinear characteristics. The Galerkin method is used to discretize the governing equations, transforming them into ordinary differential equations based on the system's mode shapes. The method of multiple scales (MMS) is then applied to these reduced equations to obtain approximate analytical solutions for the system's dynamic response. A comparison between the simplified and original equations confirms the simplified model's accuracy. The study highlights the value of advanced nonlinear analysis in enhancing the performance of capacitive energy harvesters and proposes a robust framework for their optimization using these combined methods.

**Keywords**: Electret; Method of Multiple Scales; Galerkin Method; Taylor Expansion.

## 1. Introduction

The field of energy harvesting, especially from mechanical vibrations, has gained considerable attention due to its potential in powering micro-electro-mechanical systems (MEMS) and wireless sensor networks (WSNs). Among the various energy harvesting mechanisms, electret-based capacitive harvesters stand out due to their longevity and low energy consumption. These systems employ electrets, which are dielectric materials capable of retaining electric charges over long periods, making them ideal for harvesting low-frequency environmental vibrations [1-2].

In recent years, there has been growing interest in exploiting nonlinear dynamics to enhance the efficiency of these systems. Nonlinear energy harvesters have been shown to offer a broader operational bandwidth compared to their linear counterparts, making them more effective in environments where the vibration frequency varies [3]. Techniques such as hardening and softening spring effects allow for a wider resonant frequency range, improving the system's ability to capture energy from a broader spectrum of vibrations. However, this nonlinear behavior introduces complexities in modeling and analysis, which requires robust analytical methods to accurately predict and optimize system performance.

This study focuses on the nonlinear dynamics of an electret-based capacitive energy harvester, employing semi-analytical techniques such as the method of multiple scales and Galerkin discretization. The aim is to provide a deeper understanding of the system's dynamic behavior and offer a framework for optimizing its performance. By simplifying the electrostatic forces using a third-order Taylor expansion, the complexity of the governing equations is reduced without losing critical dynamic features. The combined approach of nonlinear modeling and semi-analytical methods offers valuable insights into improving the efficiency and reliability of capacitive energy harvesters.

## 2. System characterization and analytical modeling

As illustrated in Fig. 1., the energy harvester system under study consists of a cantilever beam positioned on a fixed substrate, which is coated with an electret layer. The cantilever beam functions as the movable electrode of a variable-capacitance capacitor, forming the basis of the electrostatic energy harvesting method. The beam used in the system has a length L, thickness h, and width b. The electret layer on the fixed underlying surface is designed to generate the required voltage for harvesting environmental energy, thereby eliminating the need for any external electric charge injection into the capacitor's electrodes. Electret layer, characterized by a surface voltage of $V_s$, a thickness of $t_e$, and a relative permittivity of $\varepsilon_e$, is situated between the lower electrode and the cantilever beam.

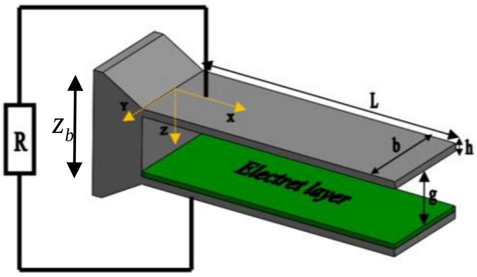
The modeling of the system is divided into two components: mechanical modeling and electrical modeling. To model the mechanical aspect of the system, we employ Hamilton's principle:

$$\int_0^t \delta(K - U + W_{ext}) = 0 \tag{1}$$

where, K, U and W represent the kinetic energy, potential energy, and work done by external forces, respectively. The potential energy of the system comprises both strain energy and electrostatic potential energy. Given the absence of external forces, the last term in Hamilton's equation reduces to zero:

$$\begin{gathered} U = U_{Electrical} + U_{Strain} \\ K = \frac{1}{2}\int_0^L \rho A \left[ \frac{\partial W}{\partial t} + \frac{dz_b}{dt} \right]^2 dx \\ W_{ext} = 0 \end{gathered} \tag{2}$$

In the above equations, $\rho$ is the mass density of the beam, A is the cross-sectional area, $Z_b$ is the base excitation and W is the deflection of beam. Furthermore, for each of the potential energy terms, we have:




$; z_b = Z_b \sin(\omega t)$

**Figure 1.** Schematic of eletret-based energy harvester

$$U_{Strain} = \frac{1}{2}\int_0^L EI\left[\frac{\partial^2 W}{\partial t^2}\right]^2 dx$$
$$U_{Electrical} = \frac{Q^2}{2C} \tag{3}$$

where, E is Young's modulus, I is the moment of inertia of the beam, Q denotes the induced electric charge on the upper electrode, while $C_{eq}$ signifies the equivalent capacitance of the system. This equivalent capacitance is the result of combining two series capacitances: the capacitance of the electret layer and the variable capacitance. To calculate the equivalent capacitance of the capacitor, considering the capacitance of the electret, we use the following equation from [4]:

$$\frac{1}{dC_{eq}} = \frac{1}{dC_{electret}} + \frac{1}{dC(t)} \;;\; dC_{electret} = \frac{\varepsilon_e \varepsilon_0 b dx}{t_e} \;,\; dC(t) = \frac{\varepsilon_0 b dx}{g - W}$$
$$C_{eq}(t) = \int_0^L \frac{\varepsilon_0 b dx}{g - W + \frac{t_e}{\varepsilon_e}} \tag{4}$$

By calculating the variation of kinetic and potential energy and substituting them into Hamilton's principle, while considering the fundamental principle of calculus of variations, the governing equation for the mechanical part of the system and the corresponding boundary conditions are obtained as follows:

$$EI\frac{\partial^4 W}{\partial x^4} + \rho A \frac{\partial^2 W}{\partial t^2} = \frac{\varepsilon_0 b Q^2}{2\left[\int_0^L \frac{\varepsilon_0 b dx}{g - W + \frac{t_e}{\varepsilon_e}}\right]^2 \left[g - W + \frac{t_e}{\varepsilon_e}\right]^2} - \rho A \ddot{z}_b \tag{5}$$

$$at\ x = 0 \;:\; W = 0 \quad \text{and} \quad \frac{\partial W}{\partial x} = 0$$
$$at\ x = L \;:\; \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 W}{\partial x^3} = 0 \tag{6}$$

To derive the governing equation for the electrical part of the system (Fig. 2.), we utilize Kirchhoff's rule and consider the equivalent electrical model of the electret as provided in the literature . The equation is given by:

$$Ri + \frac{Q}{C_{eq}} - V_s = 0 \;;\; \mathrm{i} = \frac{dQ}{dt} \tag{7}$$

where R is resistance.

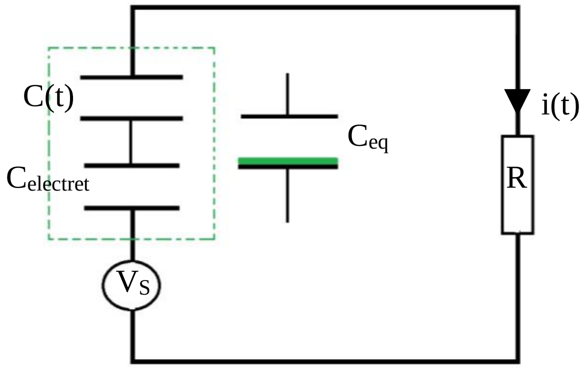



**Figure 2.** Equivalent electrical model of the energy harvester

Eqs. (5) and (7). are, in fact, the governing equations for the system's dynamics. To analyze the system's behavior, we first non-dimensionalize the equations using the following variables, which simplifies the analysis and allows for a more generalized understanding of the system's behavior across different scales:

$$\tilde{W}=\frac{W}{g}\ ,\ \tilde{x}=\frac{x}{L}\ ,\ \tilde{Q}=\frac{t_e}{\varepsilon_0\varepsilon_e bL}Q\ ,\ \tilde{\omega}=\omega\sqrt{\frac{\rho AL^4}{EI}}\ ,\ \tilde{Z}_b=Z_b\frac{\tilde{\omega}^2}{g}\ ,\ \tilde{t}=t\sqrt{\frac{EI}{\rho AL^4}} \tag{8}$$

After substituting the defined variables for non-dimensionalization and performing the necessary simplifications, we arrive at the following equations (the tilde symbol has been removed for simplicity).

$$\mathrm{H}_1=\frac{b\varepsilon_0 L^4}{2EIg^3}\ ,\ \mathrm{H}_2=1+\frac{t_e}{\varepsilon_e g}\ ,\ \mathrm{H}_3=\frac{t_e}{R\varepsilon_0\varepsilon_e bL}\sqrt{\frac{\rho AL^4}{EI}}\ ,\ \mathrm{H}_4=\frac{g}{R\varepsilon_0 bL}\sqrt{\frac{\rho AL^4}{EI}}$$

$$\begin{gathered}\frac{\partial^4 W}{\partial x^4}+\frac{\partial^2 W}{\partial t^2}=\frac{V_s^{\,2}}{(\mathrm{H}_2-1)^2}\frac{\mathrm{H}_1 Q^2}{\left[\int_0^L\frac{dx}{\mathrm{H}_2-W}\right]^2\left[\mathrm{H}_2-W\right]^2}+Z_b\sin(\omega t)\\ \frac{dQ}{dt}=\mathrm{H}_3-\mathrm{H}_4\frac{Q}{\int_0^L\frac{dx}{\mathrm{H}_2-W}}\end{gathered} \tag{9}$$

$$\begin{gathered}at\ x=0\ :\ W=0\quad\text{and}\quad\frac{\partial W}{\partial x}=0\\ at\ x=1\ :\ \frac{\partial^2 W}{\partial x^2}=0\quad\text{and}\quad\frac{\partial^3 W}{\partial x^3}=0\end{gathered} \tag{10}$$

In the next step, we will further simplify the equations using a Taylor expansion and then proceed to analyze the problem using the method of multiple time scales.

## 3. Solution strategy

As reviewed in the literature, two approaches can be considered for addressing the problem of electrostatic energy harvesters. The first method involves multiplying both sides of the first equation of the system Eq. (9). by $(\mathrm{H}_2-W)^2$ and then numerically solving the governing equations of the system, as done in [5]. This method, however, has significant computational drawbacks, including high computational cost and potential inaccuracies due to numerical approximations. The next method involves writing a Taylor series expansion for the electrostatic force in the equation and then applying semi-analytical solution methods, as performed in this paper.

### 3.1 Taylor expansion

To perform the Taylor series expansion, given that the beam's vibrations are small, the integral expression in the governing differential equations of the system will be represented as a polynomial. Given that in vibratory energy harvesters the excitation frequency is typically near the first natural frequency and seldom approaches higher mode frequencies, for MEMS harvesters, where resonance frequencies are significantly higher than environmental vibration frequencies, a single-mode response is sufficient for the analysis [6]. Assuming:

$$W(x,t) = \eta(t)\phi(x) \tag{11}$$

where $\phi(x)$ is the first mode shape of cantilever beam [7]. Now, by substituting Eq. (12). into Eq. (9)., one obtains:

$$\int_0^L \frac{dx}{\mathrm{H}_2 - W} = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 \tag{12}$$

Furthermore, the nonlinear terms present in the equations can be expressed as the following polynomial expressions:

$$\frac{1}{\left[A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3\right]^2 \left[\mathrm{H}_2 - \eta(t)\phi(x)\right]^2} = B_0(x) + B_1(x)\eta + B_2(x)\eta^2 + B_3(x)\eta^3 \tag{13}$$

$$\frac{1}{A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3} = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 \tag{14}$$

The governing differential equations of the system now take the following form:

$$\begin{gathered} \frac{d^4\phi(x)}{dx^4}\eta(t) + \phi(x)\ddot{\eta}(t) = \frac{\mathrm{H}_1 V_s^2}{(\mathrm{H}_2 - 1)^2} Q^2 (B_0(x) + B_1(x)\eta + B_2(x)\eta^2 + B_3(x)\eta^3) + Z_b \sin(\omega t) \\ \frac{dQ}{dt} = \mathrm{H}_3 - \mathrm{H}_4 Q(C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3) \end{gathered} \tag{15}$$

### 3.2 Galerkin decomposition method

After expanding the nonlinear terms in the equation, we now apply the Galerkin decomposition method to transform the equations into ordinary differential equations (ODEs):

$$\begin{gathered} \ddot{\eta}(t) + \omega_0^2\eta(t) = Q^2 V_s^2 (\psi_0 + \psi_1\eta + \psi_2\eta^2 + \psi_3\eta^3) + F\sin(\omega t) \\ \dot{Q} = \mathrm{H}_3 - \mathrm{H}_4 Q(C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3) \end{gathered} \tag{16}$$

### 3.3 Analytical solution

Since the presence of the electret in the system generates an electrostatic force, the static equilibrium position of the beam is not at $\eta = 0$. To use analytical methods, we need to separate the dynamic and static responses of the system, which can be expressed by the following relationships:

$$\begin{gathered} \eta(t) = \eta_d(t) + \eta_s \\ Q(t) = Q_d(t) + Q_s \end{gathered} \tag{17}$$

After substituting Eq. (17). into the governing equations of the system, the static equilibrium point of the system can be calculated. After substituting the equilibrium point values into the governing equations of the system and performing simplifications, we arrive at the following equations. Each coefficient is expressed as a function of the system parameters and has been written in this form to simplify the equations.

$$\begin{cases} \ddot{\eta}_d + {\omega_0}^2 \eta_d = \mu_0 Q_d + \mu_1 {Q_d}^2 + \eta_d(\mu_2 + \mu_3 Q_d + \mu_4 {Q_d}^2) + {\eta_d}^2(\mu_5 + \mu_6 Q_d + \mu_7 {Q_d}^2) + \\ \qquad {\eta_d}^3(\mu_8 + \mu_9 Q_d + \mu_{10} {Q_d}^2) + F\sin(\omega t) \\ \dot{Q}_d + \eta_d(\nu_0 + \nu_1 Q_d) + {\eta_d}^2(\nu_2 + \nu_3 Q_d) + {\eta_d}^3(\nu_4 + \nu_5 Q_d) + \nu_6 Q_d = 0 \end{cases} \tag{18}$$

The multiple scales perturbation method is applied to derive an analytical solution for the obtained equations [8]. To achieve this, the following assumptions are taken into account:

$$\begin{gathered} \eta_d = \varepsilon u \quad , \quad Q_d = \varepsilon q \\ \mu_0 = \varepsilon^2 \hat{\mu}_0 \ , \ \mu_1 = \varepsilon \hat{\mu}_1 \ , \ \mu_2 = \varepsilon^2 \hat{\mu}_2 \ , \ \mu_3 = \varepsilon \hat{\mu}_3 \ , \ \mu_5 = \varepsilon \hat{\mu}_5 \ , \ F = \varepsilon^3 \lambda \end{gathered} \tag{19}$$

by substituting Eq. (22). in Eq. (21). and removing the hat notation for brevity, one obtains:

$$\begin{gathered} \ddot{u} + \omega_0^2 u = \varepsilon^2 \mu_0 q + \varepsilon^2 \mu_1 q^2 + u(\varepsilon^2 \mu_2 + \varepsilon^2 \mu_3 q + \varepsilon^2 \mu_4 q^2) + u^2(\varepsilon^2 \mu_5 + \varepsilon^2 \mu_6 q) + u^3(\varepsilon^2 \mu_8) + \varepsilon^2 \lambda \sin(\omega t) \\ \dot{q} + u(\nu_0 + \varepsilon \nu_1 q) + u^2(\varepsilon \nu_2 + \varepsilon^2 \nu_3 q) + u^3(\varepsilon^2 \nu_4) + \nu_6 q = 0 \end{gathered} \tag{20}$$

By introducing new independent variables accordingly as $T_n = \varepsilon^n t$ ; n = 0,1, ... The derivatives with respect to t can then be expressed as a series of partial derivatives with respect to $T_n$ . The solution can be represented by expanding u and q as is done in the references [8].
Because the excitation is of order $O(\varepsilon^2)$, for consistency, $\omega - \omega_0$ must also be $O(\varepsilon^2)$. Hence, the detuning parameter $\sigma$ is defined as $\omega = \omega_0 + \varepsilon^2 \sigma$ . By substituting expanded u and q into Eq. (20). and then matching the coefficients of identical powers of $\varepsilon$, the following sets of equations are derived:

$$\varepsilon^0 : \begin{cases} D_0^2 u_0 + \omega_0^2 u_0 = 0 & \text{(a)} \\ D_0 q_0 + \nu_6 q_0 = -\nu_0 u_0 & \text{(b)} \end{cases} \tag{22}$$

$$\varepsilon^1 : \begin{cases} D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 & \text{(a)} \\ D_0 q_1 + D_2 q_0 + \nu_6 q_1 + \nu_1 u_0 q_0 + \nu_0 u_1 + \nu_2 u_0^2 = 0 & \text{(b)} \end{cases} \tag{23}$$

$$\varepsilon^2 : \begin{cases} D_0^2 u_2 + \omega_0^2 u_2 + D_1^2 u_0 + 2D_0 D_2 u_0 + 2D_0 D_1 u_1 = \mu_0 q_0 + \mu_1 q_0^2 + \mu_2 u_0 + \mu_3 u_0 q_0 + \mu_4 u_0 q_0^2 + \\ \qquad \mu_5 u_0^2 + \mu_6 u_0^2 q_0 + \mu_8 u_0^3 + \lambda \sin(\omega t) & \text{(a)} \\ D_0 q_2 + D_1 q_1 + D_2 q_0 + \nu_6 q_2 + \nu_1 u_0 q_1 + \nu_1 u_1 q_0 + \nu_0 u_2 + \nu_3 q_0 u_0^2 + 2\nu_2 u_1 u_0 + \nu_4 u_0^3 = 0 & \text{(b)} \end{cases} \tag{24}$$

The solution of Eqs. (22a) and (22b). can be written as:

$$\begin{gathered} u_0(T_0, T_1, T_2) = A(T_1, T_2) e^{i\omega_0 T_0} + \bar{A}(T_1, T_2) e^{-i\omega_0 T_0} \\ q_0(T_0, T_1, T_2) = -\nu_0 \left[ \frac{A(T_1, T_2)}{\nu_6 + i\omega_0} e^{i\omega_0 T_0} + \frac{\bar{A}(T_1, T_2)}{\nu_6 - i\omega_0} e^{-i\omega_0 T_0} \right] \end{gathered} \tag{25}$$

After substituting Eq. (25). into Eqs. (23a) and (23b). and eliminating secular terms:

$$\begin{gathered} u_1 = 0 \\ q_1 = -\frac{A^2(T_2)}{2i\omega_0 + \nu_6} e^{2i\omega_0 T_0} [\nu_2 + \frac{\nu_0}{i\omega_0 + \nu_6}] - \frac{\bar{A}^2(T_2)}{-2i\omega_0 + \nu_6} e^{-2i\omega_0 T_0} [\nu_2 + \frac{\nu_0}{-i\omega_0 + \nu_6}] - \frac{2A(T_2)\bar{A}(T_2)}{\nu_6} [\nu_2 + \frac{\nu_0}{\omega_0^2 + \nu_6^2}] \end{gathered} \tag{26}$$

Now by substituting Eqs. (26) and (25). into Eq. (24a)., The secular terms will be eliminated if:

$$\begin{gathered} -2i\omega_0 D_2 A - \frac{\nu_0 \mu_0 A}{\nu_6 + i\omega_0} + \mu_2 A + \mu_4 \nu_0^2 A^2 \bar{A} \left( \frac{2}{\omega_0^2 + \nu_6^2} + \frac{1}{(\nu_6 + i\omega_0)^2} \right) + \mu_6 \nu_0 A^2 \bar{A} \left( -\frac{2}{\nu_6 + i\omega_0} - \frac{1}{\nu_6 - i\omega_0} \right) + \\ 3\mu_8 A^2 \bar{A} + \lambda e^{i(\omega - \omega_0)T_0} = 0 \end{gathered} \tag{27}$$

Representing A in polar form:

$$A(T_2) = \frac{1}{2} a(T_2) e^{i\theta(T_2)} \tag{28}$$

Substituting into Eq. (27). and then separating the real and imaginary components one obtains:

$$\begin{cases} a\theta' = \Gamma_1 a + \Gamma_2 a^3 - \dfrac{\lambda}{\omega_0}\cos(\sigma T_2 - \theta) \\ a' = \Gamma_3 a + \Gamma_4 a^3 + \dfrac{\lambda}{\omega_0}\sin(\sigma T_2 - \theta) \end{cases} \tag{29}$$

To make the system autonomous, the $\zeta = \sigma T_2 - \theta$, change of variable can be used. By setting $a' = \zeta' = 0$ ,the frequency response for the steady-state motions can be determined as follows:

$$\sigma = \Gamma_1 + \Gamma_2 a^2 \pm \sqrt{\left(\frac{\lambda}{a\omega_0}\right)^2 - (\Gamma_3 + \Gamma_4 a^2)^2} \tag{30}$$

## 4. Conclusions

To verify the accuracy of the equations and assess whether the Taylor expansion alters the system's dynamics, we analyzed the free vibration response of the system by considering the parameters {}. As shown in Fig. 3a., the system gradually approaches its static equilibrium point after a certain period of time. This indicates that the mechanical energy generated within the system, due to the initial condition $W(1,t) = 0.15$. is eventually converted into electrical energy and dissipated through the resistor R. From Fig. 3., it is evident that the third-order Taylor expansion does not significantly affect the system's dynamics

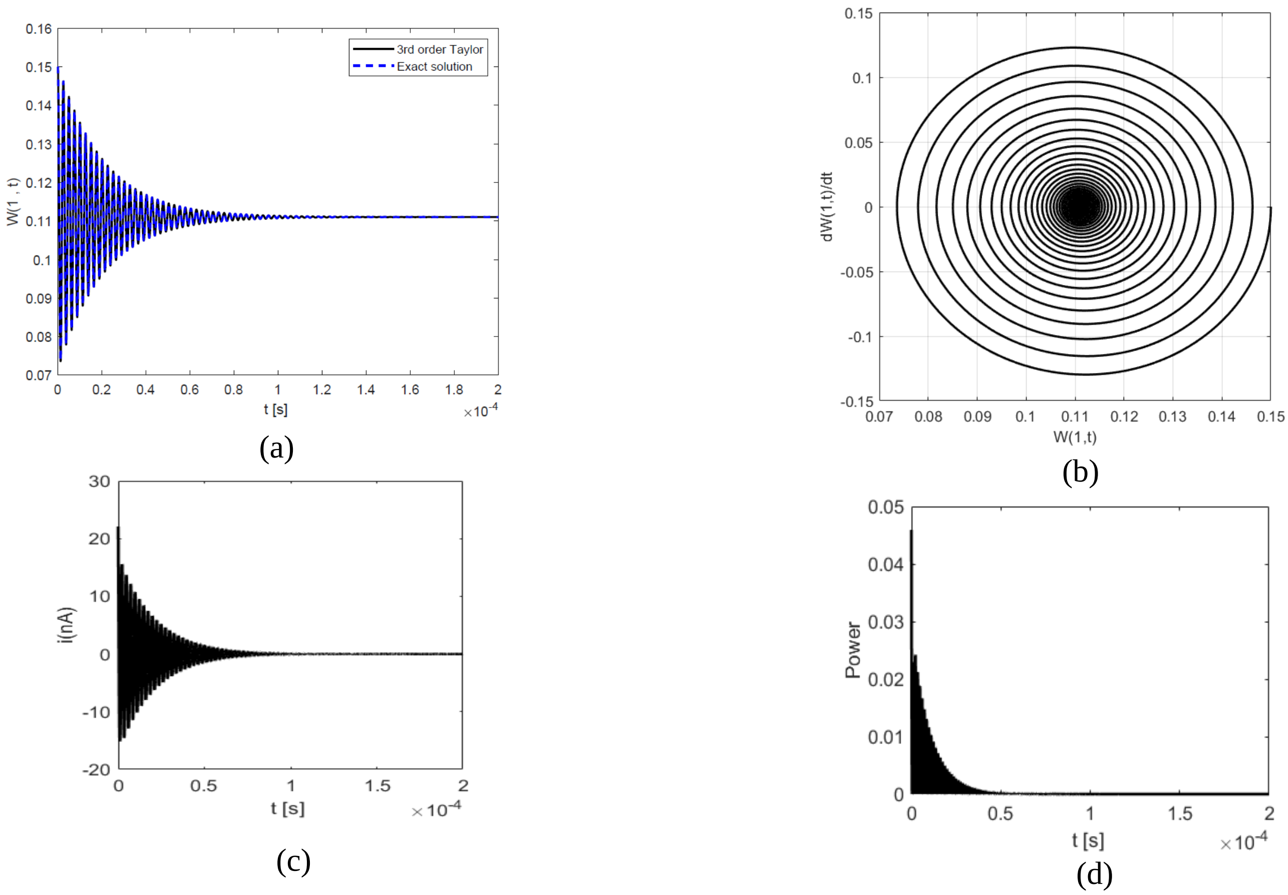
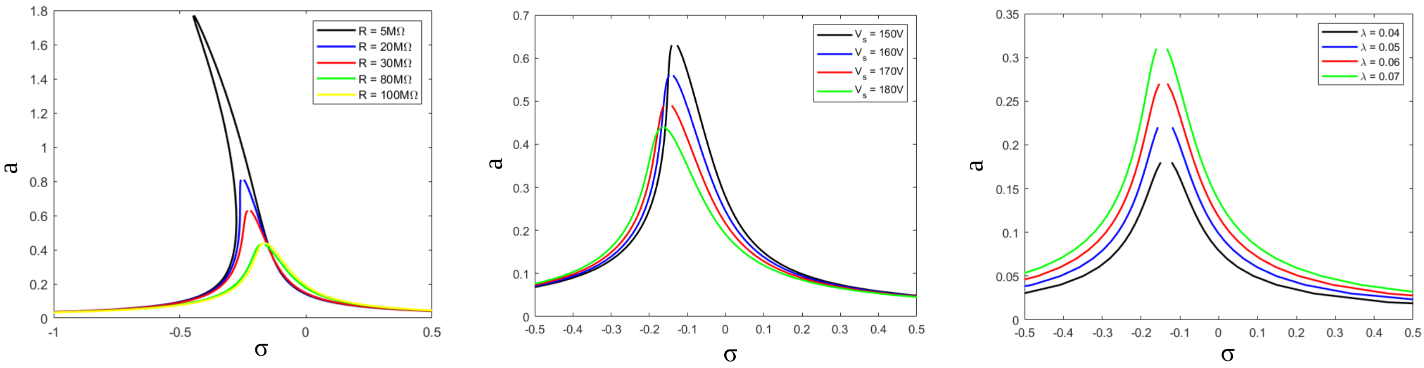



**Figure 3.** (a) Comparison of the System Dynamics After Applying the Taylor Expansion. (b) Phase portrait. (c) current. (d) Output power




(a) (b) (c)

**Figure 4.** (a) amplitude frequency response($V_S$=180v). (b amplitude frequency response(R=100 $M\Omega$). (c) amplitude frequency response($V_S$=180v and R=100 $M\Omega$).

Based on Fig. 4a., it can be observed that as the resistance R decreases, the system exhibits nonlinear behavior. This can be explained by the fact that reducing R increases the coefficient of the nonlinear term in the governing differential equations of the system. Furthermore, from the graphs in Fig. 4b., it is evident that an increase in the surface voltage of the electret layer leads to a rise in the system's resonant frequency, making the system stiffer. Additionally, Fig. 4c. clearly demonstrates that as the amplitude of the external excitation increases, the amplitude of the system's vibrations also increases.

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